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REVERSIBILITY OF FIRST-ORDER AUTOREGRESSIVE PROCESSES

Hideo ŌSAWA

Department of Mathematical Statistics, University of Electro-Communications, 1-5-1 Chofugaoka, Chofu-shi, Tokyo 182, Japan

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The present paper deals with reversibility of autoregressive processes of first order, namely AR(1). Reversibility of Markov chains with general state space was studied by Ōsawa (1985). His results are applied to AR(1) processes in this paper. Consider a Markov chain with transition densities whose state space is a Euclidian space. For reversibility of a Markov chain two ideas are available. One is reversibility with respect to some density, and the other time-reversibility. A time-reversible chain has a stationary distribution constructed by the reversible density. On the other hand a reversible chain does not necessarily have one. We first state general theorems which provide criteria for determining whether a Markov chain is reversible with respect to some density, time-reversible or not. These are applied to AR(1) processes on the real line. We shall find some examples of Markov chains which have a reversible density but cannot be time-reversible. Further, we obtain a necessary and sufficient condition under which a certain multivariate AR(1) process is reversible.

time-reversibility * Markov chains * reversible density * stationary distribution * multivariate autoregressive processes * multivariate normal distribution

1. Introduction

Reversible Markov processes have been widely applied to various stochastic phenomena, for example queues, queueing networks, storage models and so on. Following the early work of Reich (1957) for reversibility of the queue length in the $M/M/c$ with an application to a tandem queueing system, many authors have considered reversible queueing processes. In particular, Kelly (1979) developed the concept of reversibility for queueing networks. The other reversible stochastic models have been considered by Kelly (1979), Liggett (1985) and Pollett (1986). Many works on reversibility are based on birth and death processes; that is, the continuous time Markov processes with discrete state space. Further, Keilson (1979) discussed the transient behavior of a reversible Markov chain with finite state space. Kiessler (1983) investigated reversibility for flows in queueing networks as Markov chains or Markov renewal processes with discrete state space.

Recently Ōsawa (1985) dealt with reversibility of Markov chains having a general state space and its applications to some storage models. In his paper, valuable conditions for Markov chains to be time-reversible were presented. This paper applies those results to some autoregressive processes of first order, namely AR(1),

$$X_{n+1} = \alpha X_n + U_n.$$

Numerous works have been written on the statistical and probabilistic properties of autoregressive processes (for example, see Anderson (1971) and Brockwell and Davis (1987)). We are interested in reversibility of these processes.

In Section 2 we investigate two "reversibilities" of Markov chains, reversibility with respect to some density and time-reversibility, and consider the reversibility conditions. If a Markov chain is stationary, then these two notions are the same. For reversibility of Markov processes with the discrete state space, a similar treatment has been given by some authors (see Kendall (1975), Kelly (1983), Pollett (1986) and Pollett (1987)). Section 3 deals with some reversible AR(1) processes on the real line. We find the reversible models which are not time reversible. In Section 4 our main results are presented. We consider reversibility of a multivariate AR(1) process and obtain a necessary and sufficient condition for this process to be reversible with respect to some density. Further we discuss the time-reversible chain generated by such a reversible AR(1) process.

2. Reversibility of Markov chains

Let R^m be an m-dimensional Euclidean space, $\mathscr{F}$ the Borel field on R^m and l a Lebesgue measure on $(R^m, \mathscr{F})$. Consider the Markov chain $\{X_n;\ n \in N\}$ with state space R^m and transition probability $P(x, A)$, $x \in R^m$, $A \in \mathscr{F}$ where N is the set of all non-negative integers. Throughout the paper we assume that $P(x, \cdot)$ has density $p(x, y)$ with respect to l for each $x \in R^m$ and $\{Xn\}$ is l-irreducible.

The chain $\{X_n\}$ is said to be reversible with respect to a density $r(x)$ if there exists a strictly positive measurable function $r(x)$ on R^m such that

$$r(x)p(x, y) = r(y)p(y, x), \quad x, y \in R^m. \tag{2.1}$$

Such an $r(x)$ is called the reversible density for the chain $\{X_n\}$ and is determined uniquely up to constant multiples. We also say that $\{X_n\}$ has the reversible density $r(x)$.

We begin with the reversibility condition which is analogous to the case of a discrete state Markov chain.

Theorem 1.1. *Let δ be a fixed state. A Markov chain $\{X_n\}$ is reversible if and only if the transition density satisfies*

$$p(\delta, x_1)p(x_1, x_2) \cdots p(x_n, \delta) = p(\delta, x_n)p(x_n, x_{n-1}) \cdots p(x_1, \delta) \tag{2.2}$$

for any finite sequence $\{x_k\}$ in R^m.

This is shown to be similar to the discrete state case; see Kelly (1979). Here it should be noted that the reversible density is given by

$$r(x)=\frac{p(\delta, x_1)p(x_1, x_2)\cdots p(x_n, x)}{p(x, x_n)p(x_n, x_{n-1})\cdots p(x_1, \delta)}$$

where the value of $r(x)$ is independent of any finite sequence $\{x_k\}$ such that

$$p(\delta, x_1)p(x_1, x_2)\cdots p(x_n, x)>0$$

under the condition (2.2). It is not necessary to investigate (2.2) for all sequences of finite states. In fact if there exists a state δ satisfying

$$p(\delta, x)p(x, \delta)>0$$

for any $x\in R^m$, then it suffices to prove (2.2) for $n=2$; that is, the reversibility condition is written as

$$p(\delta, x)p(x, y)p(y, \delta)=p(\delta, y)p(y, x)p(x, \delta) \tag{2.3}$$

and the reversible density is given by

$$r(x)=p(\delta, x)/p(x, \delta). \tag{2.4}$$

The function $r(x)$ satisfying (2.1) is invariant; that is,

$$r(x)=\int_{R^m} r(y)p(y, x)\,\mathrm{d}y.$$

Assume that $r(x)$ is integrable; then $\{X_n\}$ has a stationary distribution π defined by

$$\pi(A)=\int_A r(x)\,\mathrm{d}x\bigg/\int_{R^m} r(x)\,\mathrm{d}x,\quad A\in\mathcal{F}. \tag{2.5}$$

Therefore the indexed set N is extended to the whole of the integers Z. Thus, the Markov chain generates the stationary chain $\{X_n;\ n\in Z\}$. Then we find that the stationary distribution π satisfies

$$\int_A \pi(\mathrm{d}x)P(x, B)=\int_B \pi(\mathrm{d}x)P(x, A) \tag{2.6}$$

for any $A, B\in\mathcal{F}$, under the condition (2.1). If the chain is stationary then (2.6) is equivalent to the following: For any $A_1, A_2, \ldots, A_s\in\mathcal{F}$, and integers $s, t, n_1, n_2, \ldots, n_s$,

$$P\left[\bigcap_{k=1}^{s}\{X_{n_k}\in A_k\}\right]=P\left[\bigcap_{k=1}^{s}\{X_{t-n_k}\in A_k\}\right] \tag{2.7}$$

(see Ōsawa (1985)). We should note that this equivalence is valid even if $P(x,\cdot)$ has no density. In general, if a stochastic process $\{X_n\}$ satisfies (2.7) then it is called time-reversible. In Kelly (1979) and Ōsawa (1985) this property is simply referred to as reversible. It is easily noted that a time-reversible Markov chain is stationary.

Moreover, for a stationary chain, (2.4) implies that a necessary and sufficient condition for $\{X_n\}$ to be time-reversible is

$$P[X_n \in A, X_{n+1} \in B] = P[X_s \in B, X_{s+1} \in A]$$

for any integers n, s and $A, B \in \mathcal{F}$. Thus time-reversibility means that $\{X_n\}$ is stochastically equivalent to the time-reversed chain $\{X_{-n}\}$.

Clearly, if a chain $\{X_n\}$ is time-reversible then it is also reversible with respect to the stationary density. However, even if a chain has a reversible density it does not necessarily have a stationary distribution. We give some examples of Markov chains having a reversible density but not any stationary distribution in the two following sections.

In the sequel we obtain the relationship between the two notions of reversibility.

Theorem 2.2. *Suppose that a Markov chain $\{X_n\}$ is reversible with respect to the density $r(x)$. If $r(x)$ is integrable and $\{X_n\}$ is stationary, then $\{X_n\}$ is time-reversible.*

From this result it follows that, for a reversible chain with respect to the density $r(x)$, if $r(x)$ is integrable then the chain is positive recurrent and if $r(x)$ is not so then it is null recurrent or transient. For Markov processes with the discrete state space, the distinction between reversibility with respect to some density and time-reversibility as above has been discussed by some authors (see Kendall (1975), Kelly (1983), Pollett (1986) and Pollett (1987)).

3. Reversible AR(1) processes on the real line

Let $\{U_n; n \in N\}$ be a sequence of independent identically distributed random variables with common probability density function $u(x)$ on the real line. Consider a process $\{X_n; n \in N\}$ defined by the stochastic difference equation

$$X_{n+1} = \alpha X_n + U_n$$

where α is a real constant. Such a $\{X_n\}$ is called an autoregressive process of first order (AR(1)). The transition probability density function is given by

$$p(x, y) = u(y - \alpha x), \quad -\infty < x, y < \infty.$$

Suppose that $u(x) > 0$ for all real x. Then, the necessary and sufficient condition for $\{X_n\}$ to be reversible with respect to some density is that for all real x and y,

$$p(0, x)p(x, y)p(y, 0) = p(0, y)p(y, x)p(x, 0),$$

or equivalently,

$$u(x)u(y - \alpha x)u(-\alpha y) = u(y)u(x - \alpha y)u(-\alpha x). \tag{3.1}$$

Then, $\{X_n\}$ has the reversible density given by

$$r(x) = u(x)/u(-\alpha x). \tag{3.2}$$

Clearly, when $\alpha=-1$ or 0, $\{X_n\}$ is reversible with respect to the density $r(x)=1$ or $u(x)$, respectively. Hence we assume that $\alpha \neq -1$ or 0 in the rest of this section.

Our main interest is in which AR(1) processes have a reversible density or stationary distribution. We first investigate the case where $u(x)$ is a two-sided exponential density; that is,

$$u(x)=\begin{cases} c\exp(-\lambda x), & x \geqslant 0, \\ c\exp(\mu x), & x<0, \end{cases}$$

where λ and μ are positive constants and $c=\lambda\mu/(\lambda+\mu)$. Suppose $\{X_n\}$ has a reversible density and set $y=\alpha x$ in (3.1). Then we have

$$u(x)u(0)u(-\alpha^2 x)=u(\alpha x)u(x-\alpha^2 x)u(-\alpha x). \tag{3.3}$$

For $0<\alpha \leqslant 1$ and $x \geqslant 0$ this relation is written as

$$\exp\{-(\lambda+\mu\alpha^2)x\}=\exp\{-(\lambda\alpha+\lambda-\lambda\alpha^2+\mu\alpha)x\}$$

and thus $\alpha=1$. Conversely, for $\alpha=1$ we can easily verify that (3.1) holds. In a similar manner we can show that (3.1) never holds for $\alpha>1$ and $\alpha<0$. Therefore, $\{X_n\}$ has a reversible density if and only if $\alpha=1$, and then from (3.2) the reversible density is

$$r(x)=\exp\{(\mu-\lambda)x\}, \quad -\infty<x<\infty.$$

However, this density cannot construct any stationary distribution.

We next consider the case where $u(x)$ is a logistic density

$$u(x)=\exp(-x)/\{1+\exp(-x)\}^2, \quad -\infty<x<\infty.$$

After some consideration it is seen that $\{X_n\}$ has the reversible density if and only if $\alpha=1$. Then, the reversible density is $r(x)=1$ and thus also in this case $\{X_n\}$ cannot generate a time-reversible chain.

We finally study the case where $u(x)$ is a normal distribution with mean μ and variance σ^2,

$$u(x)=(\sqrt{2\pi}\sigma)^{-1}\exp\{-(x-\mu)^2/(2\sigma^2)\}, \quad -\infty<x<\infty.$$

In this case $\{X_n\}$ is reversible with respect to some density for all real α. This remarkable result is shown from the fact that the function

$$\exp\{-(x-\mu)^2\}\exp\{-(y-\alpha x-\mu)^2\}\exp\{-(-\alpha y-\mu)^2\}$$

is symmetric for all real x and y. The reversible density is given by

$$\begin{aligned} r(x) &= \exp\{-(x-\mu)^2/(2\sigma^2)\}/\exp\{-(-\alpha x-\mu)^2/(2\sigma^2)\} \\ &= \exp[-\{(1-\alpha^2)x^2-2\mu(1+\alpha)x\}/(2\sigma^2)], \quad -\infty<x<\infty. \end{aligned}$$

If $|\alpha|<1$ then this density is integrable and hence $\{X_n\}$ has the stationary distribution

$$\left(\frac{1-\alpha^2}{2\pi\sigma^2}\right)^{1/2}\exp\left[-\frac{1-\alpha^2}{2\sigma^2}\left\{x-\frac{\mu}{1-\alpha}\right\}^2\right].$$

Otherwise, $r(x)$ is not integrable and is given by

$$r(x)=\begin{cases}\exp(2\mu x/\sigma^2), & \alpha=1,\\ \exp[(\alpha^2-1)\{x-\mu/(1-\alpha)\}^2/(2\sigma^2)], & |\alpha|>1\end{cases}$$

which does not construct any stationary distribution.

4. Reversibility of a multivariate AR(1)

Consider an AR(1) process $\{X_n;\, n\in N\}$ taking values in R^m; that is,

$$X_{n+1}=AX_n+U_n \tag{4.1}$$

where U_n are independent identically distributed R^m-valued random variables and A is an $m\times m$ matrix. Assume that U_n has a probability density $u(x)$, the transition density being given by

$$p(x,y)=u(y-Ax)\qquad x,y\in R^m.$$

Suppose $u(x)>0$ for all $x\in R^m$. Then $\{X_n\}$ has a reversible density if and only if

$$u(x)u(y-Ax)u(-Ay)=u(y)u(x-Ay)u(-Ax) \tag{4.2}$$

for all $x, y\in R^m$, the reversible density being given by

$$r(x)=u(x)/u(-Ax). \tag{4.3}$$

By analogy to the real-valued AR(1), if $A=-I$ or O then $\{X_n\}$ is reversible with respect to the density $r(x)=1$ or $u(x)$, respectively, where I is the identity matrix. Note that $\{X_n\}$ has no stationary distribution when $A=-I$.

In the rest of this section, we deduce the reversibility condition for an AR(1) with $u(x)$ which is a multivariate normal distribution.

Theorem 4.1. *Let $u(x)$ be a joint m-variate normal distribution $N(\mu,\Sigma)$; that is,*

$$u(x)=(2\pi)^{-m/2}|\Sigma|^{-1/2}\exp\{-(x-\mu)'\Sigma^{-1}(x-\mu)/2\},\quad x\in R^m,$$

where $\mu=(\mu_1,\mu_2,\ldots,\mu_m)'$ and Σ is the covariance matrix. Then the R^m-valued AR(1) *process defined by* (4.1) *is reversible with respect to some density if and only if the matrix $A\Sigma$ is symmetric.*

Proof. The reversibility condition (4.2) is rewritten as follows: The function

$$\begin{aligned}\exp\{-L(x,y)\}:=&\exp\{-(x-\mu)'\Sigma^{-1}(x-\mu)\}\\ &\cdot\exp\{-(y-Ax-\mu)'\Sigma^{-1}(y-Ax-\mu)\}\\ &\cdot\exp\{-(Ay+\mu)'\Sigma^{-1}(Ay+\mu)\}\end{aligned}$$

is symmetric for x and y. Therefore, it suffices to find the condition under which $L(x, y)$ is symmetric. After some basic calculations, we have

$$\begin{aligned}L(x, y) &= (x-\mu)'\Sigma^{-1}(x-\mu)+(y-\mu)'\Sigma^{-1}(y-\mu)\\&\quad+(Ax)'\Sigma^{-1}(Ax+\mu)+(Ay)'\Sigma^{-1}(Ay+\mu)\\&\quad+\mu'\Sigma^{-1}\{A(x+y)+\mu\}-x'A'\Sigma^{-1}y-y'\Sigma^{-1}Ax.\end{aligned}$$

Thus $L(x, y)$ is symmetric if and only if for all x and y,

$$x'A'\Sigma^{-1}y+y'\Sigma^{-1}Ax=y'A'\Sigma^{-1}x+x'\Sigma^{-1}Ay,$$

or equivalently,

$$x'By=y'Bx,$$

where $B=A'\Sigma^{-1}-\Sigma^{-1}A$. Hence B is a symmetric matrix. Then we have

$$\begin{aligned}A'\Sigma^{-1}-\Sigma^{-1}A &= (A'\Sigma^{-1}-\Sigma^{-1}A)'\\&=\Sigma^{-1}A-A'\Sigma^{-1}\end{aligned}$$

and thus

$$A'\Sigma^{-1}=\Sigma^{-1}A.$$

This means that $A\Sigma$ is symmetric.

Corollary 4.2. *If $A\Sigma$ is symmetric and the inverse matrix of $I-A$ exists, then the reversible* AR(1) *in Theorem* 4.1 *has the reversible density*

$$r(x)=\exp[-\{x-(I-A)^{-1}\mu\}'\Sigma^{-1}(I-A^2)\{x-(I-A)^{-1}\mu\}/2].$$

Proof. From (4.3), the reversible density is given by

$$\begin{aligned}r(x)&=\exp[-\{(x-\mu)'\Sigma^{-1}(x-\mu)-(Ax+\mu)'\Sigma^{-1}(Ax+\mu)\}/2]\\&=\exp[-\{x'(I-A'^2)\Sigma^{-1}x-x'(I+A')\Sigma^{-1}\mu-\mu'(I+A')\Sigma^{-1}x\}/2].\end{aligned}$$

Since the inverse of $I-A$ exists, this is rewritten as

$$r(x)=C\exp[-\{x-(I-A)^{-1}\mu\}'\Sigma^{-1}(I-A^2)\{x-(I-A)^{-1}\mu\}/2]$$

where C is a constant.

Remark. In this corollary, even if $I-A$ is not invertible, the reversible density exists. For example if $A=I$ then the corresponding chain is reversible with respect to

$$r(x)=\exp(x'\Sigma^{-1}\mu+\mu'\Sigma^{-1}x).$$

Corollary 4.3. *Suppose that* $\lim_{n\to\infty} A^n = O$. *Then, the reversible* AR(1) *in Theorem* 4.1 *has the stationary distribution* $N((I-A)^{-1}\mu,\ (I-A^2)^{-1}\Sigma)$.

Proof. Note that

$$(I-A^2)^{-1} = \sum_{n=0}^{\infty} A^{2n}.$$

Since $A\Sigma$ is symmetric, so are $A^n\Sigma$ and therefore $(I-A^2)^{-1}$. Further, since

$$\begin{aligned} x'A^{2n}\Sigma x &= x'A^n\Sigma(A^n)'x \\ &= \{(A^n)'x\}'\Sigma(A^n)'x > 0 \end{aligned}$$

for $x \neq 0$, $(I-A^2)^{-1}\Sigma$ is positive-definite. Thus $r(x)$ in Corollary 4.2 constructs the multivariate normal distribution with mean vector $(I-A)^{-1}\mu$ and the covariance matrix $(I-A^2)^{-1}\Sigma$.

From the above, if $A\Sigma$ is symmetric and the absolute values of all eigenvalues of A are less than unity then the definition of the process $\{X_n\}$ is extended to the set of all integers and the time-reversible chain is generated.

We close this section by studying reversibility of a bivariate AR(1). In (4.1) let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad X_n = (X_n^{(1)}, X_n^{(2)})' \quad \text{and} \quad U_n = (U_n^{(1)}, U_n^{(2)})',$$

and suppose that $u(x)$ is a bivariate normal density with mean vector and covariance matrix

$$\mu = (\mu_1, \mu_2)', \qquad \Sigma = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$$

respectively where $\sigma_1 > 0$, $\sigma_2 > 0$ and $-1 < \rho < 1$. Then, from Theorem 4.1 the reversibility condition is given by

$$\rho\sigma_1\sigma_2(a_{11} - a_{22}) + a_{12}\sigma_2^2 - a_{21}\sigma_1^2 = 0.$$

If $\rho = 0$, that is, $U_n^{(1)}$ and $U_n^{(2)}$ are independent, then we have $a_{12}\sigma_2^2 = a_{21}\sigma_1^2$ and thus

$$A = \begin{pmatrix} \alpha_1 & c\sigma_1^2 \\ c\sigma_2^2 & \alpha_2 \end{pmatrix}$$

where α_1, α_2 and c are any real constants. If $\rho \neq 0$, we get

$$a_{22} = a_{11} + (a_{12}\sigma_2^2 - a_{21}\sigma_1^2)/(\rho\sigma_1\sigma_2).$$

In particular consider an AR(1) defined by

$$X_{n+1} = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix} X_n + U_n$$

where $|\alpha_i| \neq 1$, $i = 1, 2$. If $\rho = 0$, $\{X_n\}$ has a reversible density for arbitrary real numbers α_1 and α_2. In this case $X_n^{(1)}$ and $X_n^{(2)}$ are mutually independent and the reversible density is given by

$$r(x_1, x_2) = \prod_{i=1}^{2} \exp\left[-\frac{1}{2\sigma_i^2}\{(1-\alpha_i^2)x_i^2 - 2\mu_i(1+\alpha_i)x_i\}\right].$$

If $\rho \neq 0$, the reversibility condition for $\{X_n\}$ is $\alpha := \alpha_1 = \alpha_2$ and the reversible density is given by

$$r(x) = \exp\left\{-\frac{1-\alpha^2}{2}\left(x - \frac{1}{1-\alpha}\mu\right)'\Sigma^{-1}\left(x - \frac{1}{1-\alpha}\mu\right)\right\}.$$

In each case if $|\alpha_i| < 1$, $i = 1, 2$, there exists a stationary distribution and the time-reversible chain is generated.

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